WHY A POLE IN THE CRITICAL STRIP SURVIVES THE DELAYED SECOND DIFFERENCE WITH WEIGHTS -1, 3, -2

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Abstract. Take a function g on the half-line, form three copies, delay two of them by $\log 2$ and $\log 4$, and add the copies with weights -1, 3 and -2. On the Laplace side this operation is the multiplication by the exponential polynomial $m(s) = -1+3\cdot 2^{-s}-2\cdot 4^{-s} = (1-2^{-s})(2\cdot 2^{-s}-1)$. We give a complete account of why a simple pole of the transform at a point ρ with $0 < \operatorname{Re}\rho < 1$ survives: the zeros of m are exactly the points $2\pi ik/\log 2$ and $1 + 2\pi ik/\log 2$, $k \in \mathbb{Z}$, so they lie on the two boundary lines of the strip, and the residue is multiplied by $m(\rho) \neq 0$. We add three quantitative complements: the sharp bounds $(1-2^{-\sigma})(2^{1-\sigma}-1) \le |m(\sigma+it)| \le (1+2^{-\sigma})(2^{1-\sigma}+1)$, the identification of the two boundary zeros with the compact support of the associated tent weight, and a worked example at the first zero of the Riemann zeta function, where $|m(\rho_1)| = 3.984040$. The statement is the analytic step in a delayed-difference window criterion of Crossley [1]. All statements of Sections 3–5 are checked in Lean 4 with Mathlib. Nothing here proves a zero-free region or the Riemann hypothesis.

1. Introduction

1.1. **The question.** Let $c = \log 2$. For a function $g\colon \mathbb{R} \to \mathbb{R}$ and $a \ge 0$ let

$$(\tau_a g)(u) = \begin{cases} g(u-a), & u > a, \\ 0, & u \le a, \end{cases}$$

be the copy of g delayed by a and extended by zero. The *mix* of g is

$$\begin{aligned} \mathsf{M}g &= -\tau_0 g + 3\,\tau_c\, g - 2\,\tau_{2c}\, g, \\ (\mathsf{M}g)(u) &= -g(u) + 3g(u-\log 2) - 2g(u-\log 4) \qquad (u > \log 4). \end{aligned} \tag{1}$$

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Expository companion to the film *The Mix That Cannot Kill a Pole*. The film, this paper and the Lean file `DelayedPoleSurvival.lean` were produced in one run of the Mythos chain of Math-To-Manim, with Claude Fable 5.2 as the operator of the chain. The transfer theorem (Theorem 6.2) is due to L. Crossley [1]; the proofs of Section 6 follow his Lean file. Set the author line before any submission.

Suppose that the one-sided Laplace transform $\Phi(s) = \int_0^\infty g(u)e^{-su}\,\mathrm{d}u$ continues meromorphically and has a simple pole at a point ρ with $0 < \operatorname{Re}\rho < 1$. The weights of the mix sum to zero, so the mix is built to cancel. *Does it cancel the pole?*

1.2. **The answer.** No. The Laplace transform of $\mathsf{M}g$ is $m(s)\Phi(s)$ with

$$m(s) = -1 + 3\cdot 2^{-s} - 2\cdot 4^{-s} = (1-q)(2q-1), \qquad q = 2^{-s}. \tag{2}$$

The quadratic $-1+3q-2q^2$ vanishes only at $q=1$ and $q=\frac{1}{2}$. Because $|q| = 2^{-\operatorname{Re}s}$, the strip $0 < \operatorname{Re}s < 1$ is mapped into the open annulus $\frac{1}{2} < |q| < 1$, and the two roots lie on the boundary circles of that annulus (Figure 1). Hence $m(\rho) \neq 0$, and $m\Phi$ has a simple pole at ρ with residue $m(\rho)\operatorname{Res}_{s=\rho}\Phi$.

The same computation, read in the u variable, says that each exponential $e^{\rho u}$ is an eigenfunction of the mix with eigenvalue $m(\rho)$ (Proposition 2.4). A pole of Φ at ρ is the transform's record of a term $r\,e^{\rho u}$ in g, and the mix returns $m(\rho)\,r\,e^{\rho u}$. The mix removes exactly the modes with $2^{-\rho} \in \{1, \frac{1}{2}\}$: constants, the growth e^u, and their translates by $2\pi i k/\log 2$.

1.3. **Context.** The statement is the analytic step of Crossley's delayed-difference window criterion [1]. There g is the observable attached to a family of Dirichlet L-functions, the poles of Φ sit at the zeros of the members of the family, and a Landau-type theorem from an upstream source release [3] turns a one-sided bound for a function with pole data into the absence of poles with large real part. The mix replaces a non-compact weight by a compactly supported tent (Section 5). For the criterion to remain usable, the mix must keep every pole. That is Theorem 6.2, which is `compactDifference_poleData` in [1]; a specialisation to a sum over primes is in [2].

1.4. **What this paper adds.** The theorem is not new. This paper gives its proof in ordinary mathematical language, and adds: the exact zero set of m (Proposition 3.2); sharp lower and upper bounds for $|m|$ on vertical lines (Proposition 3.3), which bound the change of each residue (Theorem 4.2); the remark that the two boundary zeros are forced by the compact support of the tent (Proposition 5.2); an example of a two-term mix that *does* remove poles on the critical line (Remark 4.4); and a numerical example (Section 7). Section 8 lists the Lean names of the statements.

2. Notation and the transform of the mix

Throughout, $c = \log 2$, $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$, and $q = q(s) = e^{-sc} = 2^{-s}$. For measurable $g\colon \mathbb{R} \to \mathbb{C}$ and $s \in \mathbb{C}$ with $\int_0^\infty |g(u)|e^{-\sigma u}\,\mathrm{d}u < \infty$ we write $\mathcal{L}g(s) = \int_0^\infty g(u)e^{-su}\,\mathrm{d}u$.

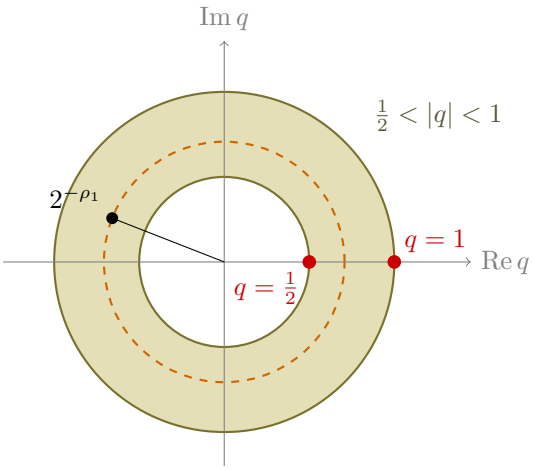


FIGURE 1. The q-plane, $q = 2^{-s}$. The strip $0 < \operatorname{Re} s < 1$ is mapped into the shaded open annulus. The two zeros of $(1-q)(2q-1)$ lie on its boundary circles. The dashed circle $|q| = 2^{-1/2}$ is the image of the critical line; the black point is $2^{-\rho_1}$ for the first zeta zero $\rho_1 = \frac{1}{2} + 14.134725\,i$.

Lemma 2.1 (A delay is a multiplication)**.** *Let $a \geq 0$ and let g be measurable with $\int_0^\infty |g(u)|e^{-\sigma u}\,\mathrm{d}u < \infty$. Then $\int_0^\infty |\tau_a g(u)|e^{-\sigma u}\,\mathrm{d}u < \infty$ and*

$$\mathcal{L}(\tau_a g)(s) = e^{-sa}\,\mathcal{L}g(s).$$

Proof. Since $\tau_a g$ vanishes on $(0, a]$, the substitution $u = v + a$ gives

$$\begin{aligned}\int_0^\infty \tau_a g(u)e^{-su}\,\mathrm{d}u &= \int_a^\infty g(u-a)e^{-su}\,\mathrm{d}u \\ &= \int_0^\infty g(v)e^{-s(v+a)}\,\mathrm{d}v = e^{-sa}\mathcal{L}g(s).\end{aligned}$$

The same substitution with absolute values gives the integrability, with the factor $e^{-\sigma a} \leq 1$. □

Proposition 2.2 (The mix is a multiplication)**.** *Let g be measurable with $|g(u)| \leq Ce^{2u}$ for $u \geq 0$. Then $|\mathsf{M}g(u)| \leq 6Ce^{2u}$ for $u \geq 0$, and for $\operatorname{Re} s > 2$*

$$\mathcal{L}(\mathsf{M}g)(s) = m(s)\,\mathcal{L}g(s), \qquad m(s) = -1 + 3e^{-sc} - 2e^{-2sc} = (1-q)(2q-1).$$

Proof. For $a \geq 0$ and $u > a$ we have $|\tau_a g(u)| \leq Ce^{2(u-a)} \leq Ce^{2u}$, and $\tau_a g(u) = 0$ for $u \leq a$; this gives the bound with $1 + 3 + 2 = 6$. For $\operatorname{Re} s > 2$ all three integrals converge absolutely, so Lemma 2.1 and linearity give $\mathcal{L}(\mathsf{M}g) = (-1 + 3e^{-sc} - 2e^{-2sc})\mathcal{L}g$. Finally $(1-q)(2q-1) = 2q - 1 - 2q^2 + q = -1 + 3q - 2q^2$ and $e^{-2sc} = q^2$. □

Remark 2.3 (Convolution form). If g vanishes on $(-\infty, 0]$, then $\mathsf{M}g = g * \mu$ with the measure $\mu = -\delta_0 + 3\delta_c - 2\delta_{2c}$, and $m(s) = \int e^{-sa}\,\mathrm{d}\mu(a)$ is the Laplace transform of μ. So m is an exponential polynomial with the three

frequencies 0, c, $2c$. They are commensurable; this is why m is a polynomial in the single variable q and why its zero set is a union of two arithmetic progressions on two vertical lines.

Proposition 2.4 (Exponentials are eigenfunctions)**.** *For $\lambda \in \mathbb{C}$ put $e_\lambda(u) = e^{\lambda u}$. Then for $u > 2c$,*

$$(\mathsf{M}e_\lambda)(u) = m(\lambda)\, e^{\lambda u}.$$

In particular M annihilates the constants and the function e^u on $(2c, \infty)$: $m(0) = -1 + 3 - 2 = 0$ and $m(1) = -1 + \frac{3}{2} - \frac{2}{4} = 0$.

Proof. For $u > 2c$ all three delays have started, so $(\mathsf{M}e_\lambda)(u) = -e^{\lambda u} + 3e^{\lambda(u-c)} - 2e^{\lambda(u-2c)} = (-1 + 3e^{-\lambda c} - 2e^{-2\lambda c})e^{\lambda u}$. □

3. The symbol and its zeros

Lemma 3.1. *$|q(s)| = 2^{-\sigma}$ and $\arg q(s) \equiv -t \log 2 \pmod{2\pi}$. If $0 < \sigma < 1$ then $\frac{1}{2} < |q(s)| < 1$.*

Proof. $q = e^{-(\sigma+it)c} = e^{-\sigma c} e^{-itc}$. The function $\sigma \mapsto 2^{-\sigma}$ is strictly decreasing with values 1 at $\sigma = 0$ and $\frac{1}{2}$ at $\sigma = 1$. □

Proposition 3.2 (The zero set)**.** *$m(s) = 0$ if and only if*

$$s = \frac{2\pi i k}{\log 2} \quad \text{or} \quad s = 1 + \frac{2\pi i k}{\log 2} \qquad \text{for some } k \in \mathbb{Z}.$$

All zeros are simple. In particular every zero of m has real part 0 or 1, and $m(s) \neq 0$ for $0 < \operatorname{Re} s < 1$.

Proof. By (2), $m(s) = 0$ if and only if $q = 1$ or $q = \frac{1}{2}$. Now $e^{-sc} = 1$ if and only if $-sc \in 2\pi i\mathbb{Z}$, and $e^{-sc} = \frac{1}{2} = e^{-c}$ if and only if $-sc + c \in 2\pi i\mathbb{Z}$. This gives the two progressions. For the multiplicity, $\frac{\mathrm{d}}{\mathrm{d}s} q = -cq$, hence

$$m'(s) = -cq(-(2q-1) + 2(1-q)) = -cq\,(3 - 4q), \tag{3}$$

which equals c at $q = 1$ and $-\frac{c}{2}$ at $q = \frac{1}{2}$. The last assertion also follows from Lemma 3.1: inside the strip $|q| \neq 1$ and $|q| \neq \frac{1}{2}$. □

The spacing of the zeros on each line is $2\pi/\log 2 = 9.064720\ldots$.

Proposition 3.3 (Sharp bounds on vertical lines)**.** *Let $0 \le \sigma \le 1$ and put $x = 2^{-\sigma} \in [\frac{1}{2}, 1]$. For every $t \in \mathbb{R}$,*

$$\varphi(\sigma) := (1 - x)(2x - 1) \;\le\; |m(\sigma + it)| \;\le\; (1 + x)(2x + 1). \tag{4}$$

Equality holds on the left exactly when $t \in \frac{2\pi}{\log 2}\mathbb{Z}$, and on the right exactly when $t \in \frac{2\pi}{\log 2}(\mathbb{Z} + \frac{1}{2})$. The function φ is positive on $(0, 1)$, vanishes at 0 and 1, and has its maximum $\frac{1}{8}$ at $\sigma = \log_2 \frac{4}{3} = 0.415037\ldots$. On the critical line, $\varphi(\frac{1}{2}) = \frac{3}{\sqrt{2}} - 2 = 0.121320\ldots$ and the upper bound is $\frac{3}{\sqrt{2}} + 2 = 4.121320\ldots$.

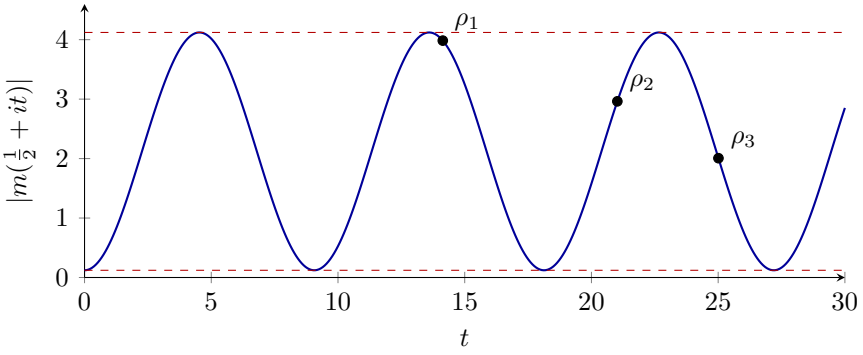


FIGURE 2. The modulus of the symbol on the critical line. It is periodic with period $2\pi/\log 2 = 9.0647$ and stays between the dashed lines $\frac{3}{\sqrt{2}} \mp 2$ of Proposition 3.3. The marked points are the first three zeros of ζ.

Proof. Write $q = xe^{i\theta}$ with $\theta = -t\log 2$. By the triangle inequality, $1 - x \le |1 - q| \le 1 + x$ and $2x - 1 \le |2q - 1| \le 2x + 1$; note that $1 - x \ge 0$ and $2x - 1 \ge 0$. Multiplying gives (4). In $|1 - q| \ge 1 - |q|$ equality holds if and only if $q \ge 0$, and in $|2q - 1| \ge 2|q| - 1$ (with $2|q| \ge 1$) equality holds if and only if $q \ge 0$. Since $x > 0$, both hold exactly when $e^{i\theta} = 1$. The two upper inequalities are equalities exactly when $e^{i\theta} = -1$. For the maximum, $(1 - x)(2x - 1) = -2x^2 + 3x - 1$ is maximal at $x = \frac{3}{4}$ with value $\frac{1}{8}$. At $x = 2^{-1/2}$, $(1 - x)(2x - 1) = 3x - 1 - 2x^2 = \frac{3}{\sqrt{2}} - 2$ and $(1 + x)(2x + 1) = 3x + 1 + 2x^2 = \frac{3}{\sqrt{2}} + 2$. □

Corollary 3.4 (Real points)**.** *If $0 < \sigma < 1$ then $m(\sigma) = \varphi(\sigma) > 0$.*

Proof. For real $s = \sigma$ the number $q = 2^{-\sigma}$ is real, with $\frac{1}{2} < q < 1$, so both factors of $(1 - q)(2q - 1)$ are positive. □

4. The pole survives

Lemma 4.1 (An analytic multiplier and a simple pole)**.** *Let $\rho \in \mathbb{C}$, let h and M be analytic in a neighbourhood U of ρ, let $r \in \mathbb{C}$, and let $\Phi(s) = \dfrac{r}{s - \rho} + h(s)$ on $U \setminus \{\rho\}$. Then on $U \setminus \{\rho\}$*

$$M(s)\Phi(s) = \frac{M(\rho)\, r}{s - \rho} + \tilde{h}(s), \qquad \tilde{h}(s) = M(s)h(s) + r\,\frac{M(s) - M(\rho)}{s - \rho},$$

and $\tilde{h}$ extends analytically to ρ, with $\tilde{h}(\rho) = M(\rho)h(\rho) + r\,M'(\rho)$. Consequently $M\Phi$ has at ρ a simple pole with residue $M(\rho)r$ if $M(\rho)r \neq 0$, and a removable singularity otherwise.

Proof. The identity is $M(s)r = M(\rho)r + r\,(M(s) - M(\rho))$ divided by $s - \rho$. The difference quotient $(M(s) - M(\rho))/(s - \rho)$ has a removable singularity at ρ with value $M'(\rho)$. □

Theorem 4.2 (The pole survives the mix)**.** *Let $0 < \operatorname{Re}\rho < 1$, let h be analytic near ρ, $r \neq 0$, and $\Phi(s) = \dfrac{r}{s-\rho} + h(s)$ near ρ. Then $m\Phi$ has a simple pole at ρ, with*

$$\operatorname*{Res}_{s=\rho} m\Phi = m(\rho)\, r \neq 0,$$

$$\varphi(\operatorname{Re}\rho)\,|r| \;\le\; |m(\rho)\, r| \;\le\; \big(1 + 2^{-\operatorname{Re}\rho}\big)\big(2^{1-\operatorname{Re}\rho} + 1\big)|r|.$$

If ρ is real and $r > 0$, then $m(\rho)\, r > 0$.

Proof. The symbol m is entire, so Lemma 4.1 applies with $M = m$. By Proposition 3.2, $m(\rho) \neq 0$. The bounds are Proposition 3.3 and the last statement is Corollary 3.4. □

Remark 4.3 (What happens on the walls)*.* If Φ has a simple pole at a zero of m, for instance at $s = 0$ or $s = 1$, then by Lemma 4.1 the product $m\Phi$ is analytic there: the mix removes that pole. By Proposition 2.4 this is the statement that the mix annihilates the corresponding mode. A double pole at a zero of m becomes a simple pole, because the zeros of m are simple.

Remark 4.4 (A mix that does kill)*.* The conclusion of Theorem 4.2 is a property of the weights. The two-term mix $g \mapsto \tau_0 g - \sqrt{2}\,\tau_c g$ has the symbol $1 - \sqrt{2}\,q$, which vanishes when $q = 2^{-1/2}$, that is at $s = \frac{1}{2} + 2\pi i k/\log 2$, $k \in \mathbb{Z}$. These points are on the critical line, and a pole of Φ at one of them would be removed. In general a mix with weights w_j and delays jc has the symbol $P(q) = \sum_j w_j q^j$, and a simple pole at ρ survives if and only if $P(2^{-\rho}) \neq 0$. Up to a scalar, $-1 + 3q - 2q^2 = -2(q-1)(q-\frac{1}{2})$ is the only quadratic whose roots are 1 and $\frac{1}{2}$. Section 5 explains why these two roots are needed.

5. What the weights are for: the tent

Let $x_+ = \max(x, 0)$. For $t > 0$ define

$$W(t) = \Big(\frac{4}{t} - 1\Big)_+ - 3\Big(\frac{2}{t} - 1\Big)_+ + 2\Big(\frac{1}{t} - 1\Big)_+. \tag{5}$$

Lemma 5.1 (The tent)**.** *$W(t) = 0$ for $t \le 1$ and for $t \ge 4$, $W(t) = 2\big(1 - \frac{1}{t}\big)$ for $1 < t \le 2$, and $W(t) = \frac{4}{t} - 1$ for $2 < t < 4$. So W is continuous, non-negative, supported in $[1, 4]$, with $W(2) = 1$.*

Proof. For $t \le 1$ all three brackets are non-negative and $(\frac{4}{t} - 1) - 3(\frac{2}{t} - 1) + 2(\frac{1}{t} - 1) = \frac{4-6+2}{t} + (-1 + 3 - 2) = 0$. For $1 < t \le 2$ the third bracket is negative and $(\frac{4}{t} - 1) - 3(\frac{2}{t} - 1) = 2 - \frac{2}{t}$. For $2 < t < 4$ only the first bracket is positive. For $t \ge 4$ none is. □

The cancellation for $t \le 1$ uses both relations $-1 + 3 - 2 = 0$ and $4 - 6 + 2 = 0$; in the variable q these are $m(0) = 0$ and $m(1) = 0$ after multiplication by -1 and by -4.

Proposition 5.2 (Compact support forces the two boundary zeros)**.** *For every $s \in \mathbb{C} \setminus \{0, 1\}$,*

$$W^*(s) := \int_0^\infty W(t)\, t^{s-1}\, \mathrm{d}t = -\frac{4^s\, m(s)}{s(s-1)}.$$

The left side is an entire function of s, so the zeros of m at $s = 0$ and $s = 1$ are exactly what cancels the two poles of $1/(s(s-1))$. Moreover $W^*(1) = \int_1^4 W(t)\, \mathrm{d}t = 2 \log 2$.

Proof. For $a > 0$ and $\operatorname{Re} s > 1$, $\int_0^\infty (\frac{a}{t} - 1)_+ t^{s-1}\, \mathrm{d}t = \int_0^a (at^{s-2} - t^{s-1})\, \mathrm{d}t = \frac{a^s}{s-1} - \frac{a^s}{s} = \frac{a^s}{s(s-1)}$. Hence, for $\operatorname{Re} s > 1$, $W^*(s) = \dfrac{4^s - 3 \cdot 2^s + 2}{s(s-1)} = \dfrac{4^s(1 - 3 \cdot 2^{-s} + 2 \cdot 4^{-s})}{s(s-1)} = -\dfrac{4^s m(s)}{s(s-1)}$. By Lemma 5.1, W is bounded with support in $[1, 4]$, so W^* is entire, and the identity extends to $\mathbb{C} \setminus \{0, 1\}$ by analytic continuation. At $s = 1$, by (3), $m'(1) = -\frac{c}{2}$, so the right side tends to $-4m'(1) = 2 \log 2$. Directly, $\int_1^2 2(1 - \frac{1}{t})\, \mathrm{d}t + \int_2^4 (\frac{4}{t} - 1)\, \mathrm{d}t = (2 - 2 \log 2) + (4 \log 2 - 2) = 2 \log 2$. □

Example 5.3 (The zeta case)**.** Let Λ be the von Mangoldt function and $R(y) = \sum_n \Lambda(n) (\frac{y}{n} - 1)_+$. By (5), with $t = n/y$,

$$A(y) := R(4y) - 3R(2y) + 2R(y) = \sum_{y < n < 4y} \Lambda(n)\, W\Big(\frac{n}{y}\Big) :$$

the three-term difference of ramps is a sum over the window $(y, 4y)$ only. Following [1], let $G(y) = R(y) - y \log y + (1 + \gamma)y$, where γ is Euler's constant. A direct computation gives

$$-G(4y) + 3G(2y) - 2G(y) = 2y \log 2 - A(y),$$

because $4y \log(4y) - 6y \log(2y) + 2y \log y = 2y \log 2$ and $4y - 6y + 2y = 0$. With $g(u) = G(e^u)$ and $u > 2c$ this reads $(\mathsf{M}g)(u) = 2y \log 2 - A(y)$ at $y = e^u/4$: the mix of g is the main term $W^*(1)\, y$ minus a compactly supported weighted count of prime powers.

For the transform, $\int_1^\infty (\frac{y}{n} - 1)_+ y^{-s-1}\, \mathrm{d}y = \frac{n^{-s}}{s(s-1)}$ for $\operatorname{Re} s > 1$, so

$$\begin{aligned} \Phi(s) &:= \int_0^\infty g(u) e^{-su}\, \mathrm{d}u \\ &= -\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{1}{s(s-1)} - \frac{1}{(s-1)^2} + \frac{1 + \gamma}{s - 1} \qquad (\operatorname{Re} s > 1). \end{aligned}$$

Since $-\zeta'/\zeta(s) = \frac{1}{s-1} - \gamma + O(s-1)$ and $\frac{1}{s(s-1)} = \frac{1}{s-1} - 1 + O(s-1)$, the right side is analytic at $s = 1$: this is the role of the two correction terms in G. It is meromorphic in $\mathbb{C}$, and at a zero ρ of ζ of multiplicity k_ρ with $0 < \operatorname{Re} \rho < 1$ it has a simple pole with residue $k_\rho/(\rho(1 - \rho))$; see [4, §12] for the partial-fraction expansion of ζ'/ζ. By Theorem 4.2 the transform $m\Phi$ of $\mathsf{M}g$ has the same poles in the strip, with residues $m(\rho)k_\rho/(\rho(1 - \rho))$.

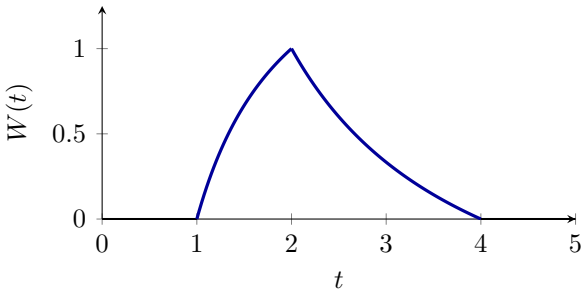


FIGURE 3. The tent weight W of Lemma 5.1. Its integral is $2\log 2$.

The simple pole of Φ at $s = 0$, with residue $\zeta'(0)/\zeta(0) = \log 2\pi$, is removed (Remark 4.3).

6. The transfer of pole data

The upstream criterion works with the following package of hypotheses. It is the structure `PoleData` of [3], copied in `DelayedPoleSurvival.lean`.

Definition 6.1 (Pole data)**.** Let $g\colon \mathbb{R} \to \mathbb{R}$, $\Phi\colon \mathbb{C} \to \mathbb{C}$ and $Z \subset \mathbb{C}$. The triple (g, Φ, Z) is *pole data* if:

(1) g is measurable, and $|g(u)| \le Ce^{2u}$ for all $u \ge 0$, for some C;
(2) $\Phi(s) = \int_0^\infty g(u)e^{-su}\,\mathrm{d}u$ for $\operatorname{Re} s > 2$;
(3) every $\rho \in Z$ has $\operatorname{Re}\rho > 0$;
(4) Φ is complex differentiable on $\{\operatorname{Re} s > 0\} \setminus Z$ (in the sense of differentiability within that set);
(5) for every z with $\operatorname{Re} z > 0$ there is a punctured neighbourhood of z disjoint from Z;
(6) for every $\rho \in Z$ there are $r \ne 0$ and h analytic at ρ with $\Phi(s) = \frac{r}{s-\rho} + h(s)$ on a punctured neighbourhood of ρ; and if ρ is real then $r > 0$.

Theorem 6.2 (Crossley [1]; `compactDifference_poleData`)**.** *Let (g, Φ, Z) be pole data and assume $\operatorname{Re}\rho < 1$ for every $\rho \in Z$. Then $(\mathsf{M}g,\ m\Phi,\ Z)$ is pole data. For each $\rho \in Z$ the new residue is $m(\rho)$ times the old one.*

Proof. (1) Each $\tau_a g$ is measurable, as the product of the indicator of (a, ∞) with a translate of g; the growth bound, with the constant $6C$, is in Proposition 2.2. (2) is Proposition 2.2. (3) and (5) concern Z only. (4) holds because m is entire. (6) For $\rho \in Z$ we have $0 < \operatorname{Re}\rho < 1$, so Theorem 4.2 gives the expansion of $m\Phi$ with residue $m(\rho)r \ne 0$, and $m(\rho)r > 0$ when ρ is real. □

Remark 6.3 (Why the poles are needed)*.* A classical theorem of Landau states that the Laplace transform of a function of one sign is analytic on the real axis to the right of a real singularity that it must possess at its abscissa

of convergence; see [5, §15.1]. Criteria of Landau type use this to pass from a one-sided bound for g to the absence of singularities of Φ with large real part. In the source release [3], the theorem `landau_window` has this form, as it is used in [1]: for pole data (g, Φ, Z), $0 < \theta < 1$ and $\delta > 0$, a one-sided window hypothesis on g with exponent θ implies $\operatorname{Re}\rho \le \theta$ for all $\rho \in Z$. Theorem 6.2 allows one to apply it to $\mathsf{M}g$ in place of g with the same set Z. If the mix had removed a pole, the conclusion for that point of Z would have been lost. We do not prove or use the Landau window theorem here.

7. A worked example

Let $\rho_1 = \frac{1}{2} + i\gamma_1$, $\gamma_1 = 14.134725141734\ldots$, be the first zero of ζ in the upper half-plane; it is a simple zero. All values below are rounded to six decimals.

quantity	value
$q = 2^{-\rho_1}$	$-0.658571 + 0.257458\,i$
$\lvert q\rvert = 2^{-1/2}$, $\arg q$	0.707107, 158.648°
$1 - q$	$1.658571 - 0.257458\,i$
$2q - 1$	$-2.317141 + 0.514916\,i$
$m(\rho_1) = (1-q)(2q-1)$	$-3.710574 + 1.450591\,i$
$\lvert m(\rho_1)\rvert$	3.984040
bounds of Proposition 3.3 at $\sigma = \frac{1}{2}$	$0.121320 \le \lvert m\rvert \le 4.121320$
$\operatorname{Res}_{\rho_1}\Phi = 1/(\frac{1}{4} + \gamma_1^2)$ (Example 5.3)	0.004999
$\lvert\operatorname{Res}_{\rho_1} m\Phi\rvert$	0.019916

The angle is $-\gamma_1 \log 2 = -9.797445$ radians, which is 158.648° modulo 360°. So q is close to the negative real axis, where $\lvert m\rvert$ is largest, and the residue at ρ_1 grows by a factor close to 4. For the next two zeros, $\gamma_2 = 21.022040$ and $\gamma_3 = 25.010858$, the factors are $\lvert m(\rho_2)\rvert = 2.962691$ and $\lvert m(\rho_3)\rvert = 2.006492$ (Figure 2). The smallest possible factor on the critical line is 0.121320, attained only at the points $\frac{1}{2} + 2\pi i k/\log 2$; whether a zero of ζ lies near such a point is a question about $\gamma \log 2$ modulo 2π, on which nothing is claimed here.

In the film these numbers appear as follows. The delayed tone is "turned by 159 degrees and shrunk to 0.707"; the three arrows -1, $3q$, $-2q^2$ placed head to tail leave a gap of length 3.98; the surface $\lvert m\rvert$ over the strip has a pillar of height 3.984 at ρ_1; and the inset graph is φ, with the value 0.121 at $\sigma = \frac{1}{2}$.

8. Formal verification

The file `DelayedPoleSurvival.lean` (namespace `PrimeWindow`) imports only Mathlib. It was checked with Lean 4.33.1 and Mathlib revision `0df444a`, the versions pinned by [1]. Every theorem depends only on the axioms `propext`, `Classical.choice` and `Quot.sound`.

statement	Lean name (our file)	in [1]
Lemma 2.1	`laplace_delay`	`laplace_delay`
Proposition 2.2	`laplace_mix`, `symbol_expand`	inside the main proof
Proposition 2.4, real λ	`mix_exp`, `mix_const`, `mix_exp_log_two`	—
Lemma 3.1	`norm_q`, `q_mem_annulus`	inside `symbol_strip`
Proposition 3.2	`symbol_ne_zero`, `symbol_zero_re`, `symbol_zero_left`, `symbol_zero_right`	`symbol_strip`
Proposition 3.3, lower bound	`symbol_norm_lower`, `floor_pos`	—
Corollary 3.4	`symbol_pos_of_real`	`symbol_real`
Lemma 4.1, entire M	`pole_transfer`	inside the main proof
Theorem 4.2	`pole_survives_mix`, `residue_norm_lower`	—
Lemma 5.1	`tent_eq`	—
Theorem 6.2	`mix_poleData`	`compactDifference_poleData`

Not formalised in our file: the upper bound and the equality cases of Proposition 3.3; the simplicity of the zeros and the "only if" direction of Proposition 3.2 in its exact form (the file proves that every zero has real part 0 or 1, and that the two progressions consist of zeros); Proposition 5.2; Example 5.3. The file has a floating-point layer that evaluates q, $m(\rho_1)$, $\varphi(\frac{1}{2})$ and the minimum of $|m|/\varphi$ on a grid of the strip, and compares them with the values of Section 7; this layer is a test, not a proof.

9. Scope

This paper proves that the mix (1) preserves simple poles in the open strip, with explicit control of the residues. It does not prove the Landau window theorem of [3], it does not verify that a particular arithmetic function satisfies Definition 6.1, and it does not prove a zero-free region. The criteria of [1, 2] are conditional statements: they derive a conclusion about zeros from a hypothesis on windows that is not known to hold. Neither those criteria nor this paper prove the Riemann hypothesis.

References

[1] L. Crossley, *A delayed-difference window criterion for Dirichlet families*, Lean 4 source and written proof, directory `results/dirichlet-window-transfer` of the repository `https://github.com/leecrossley/prime-window-rh-criterion`, 2026. DOI 10.5281/zenodo.24410862. Licence CC BY 4.0.

[2] L. Crossley, *An ordinary-prime window criterion for the Riemann hypothesis*, Lean 4 source and written proof, `https://github.com/leecrossley/prime-window-rh-criterion`, 2026. DOI 10.5281/zenodo.24410878.

[3] Lean source release pinned by [1] (file `LandauZeroFreeCriterion.lean`, definitions `PoleData` and `WindowHypU`); readable mirror at `https://github.com/Deicyde/riemann-hypothesis`, commit `40e56ba`.

[4] H. Davenport, *Multiplicative Number Theory*, 3rd ed., revised by H. L. Montgomery, Graduate Texts in Mathematics 74, Springer, 2000.

[5] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge Studies in Advanced Mathematics 97, Cambridge University Press, 2007.

[6] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381.